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Bounds on Multibunch Growth Rates when the Bunches Currents are not Identical

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Abstract

In [1], it is demonstrated that for a partially filled ring (where several bunches are missing from a symmetric filling) one can find a bound on the growth rate of instabilities and demonstrate that if a feedback system stabilizes the system for a symmetric filling, the system is stable for a partial filling as well. Here, this result is refined to include the case where all the bunches have different currents ([1] assumes that the non-empty buckets all had the same charge). In addition, the limitations of the result (and hence of [1]) are described.

1 Introduction

It is fairly straightforward to find the complex eigenfrequencies of a multibunch accelerator when all the bunches contain the same amount of charge and are equally spaced (a symmetric filling). However, in most multibunch machines, this is not the real situation. It is a good approximation that the bunches all lie in equally spaced "buckets," but not all of these buckets actually contain a bunch. In principle the bunches will all contain slightly different charges. Solving the eigensystem in this case is a much more formidable task: the size of the eigensystem solved for a symmetric filling is multiplied by (at least) the number of bunches, which can be an extremely large number (the LHC has over 2800 bunches and PEP-II has over 1600 bunches).

In [1], Kohaupt makes a significant contribution to the solution of this problem by demonstrating that one can use the growth rates computed for the symmetric filling to give bounds on the growth rates when several bunches are removed from a symmetric filling. In particular, the largest growth rate for a symmetric filling gives a bound for the growth rates for this partial filling.

This paper expands on the result of [1] by considering the case where the buckets can contain any amount of charge. In addition, the proof in [1] is improved upon by indicating more precisely assumptions required for validity, and demonstrating an important algebraic property of the system (see section 3.2) needed for the validity of this proof and that in [1].

2 Construction of the Algebraic System

For an accelerator with a linear lattice (partially including first-order chromatic effects), where each bunch is centered in one of many equally space "buckets," one can write down an eigenvalue equation describing the system. A simplified form of this equation is [2, 3, 4]

$$\frac{\Omega - \boldsymbol{m}\cdot\boldsymbol{\omega}}{\omega_0}\Psi_{r\boldsymbol{m}b} = -i\frac{q^2\beta}{\gamma m_q c L}\sum_{a\boldsymbol{m}'np} N_n e^{2\pi i p(r-n)/M} Z(\omega_p) k_{\boldsymbol{m}b}(\omega_p) h_{\boldsymbol{m}'a}(\omega_p)\Psi_{n\boldsymbol{m}'a}. \tag{1}$$

The symbols are given by

a,b	Radial mode indices, describing the phase-space dependence of the basis functions in J
c	The speed of light
$h_{\boldsymbol{m}b}(\omega)$, $k_{\boldsymbol{m}b}(\omega)$	Fourier transform of the phase-space basis functions (indexed by $\boldsymbol{m}$ and b) which are used to expand perturbations to the distribution. There are also some ω-independent constants mixed in.
L	The length of the ring
M	The number of buckets.
$\boldsymbol{m}$,$\boldsymbol{m}'$	Azimuthal modes indices, such that the basis function has a phase-space dependence of $e^{i\boldsymbol{m}\cdot\boldsymbol{\theta}}$
m_q	Mass per particle
N_n	Number of particles in bucket n
n,r	Indices for bucket number
q	Charge per particle
$Z(\omega)$	The impedance, more or less. In the transverse case, this would be $\beta_\perp Z_\perp(\omega)$, in the longituidinal case, this would be $\omega_0 Z_\parallel(\omega)/\omega$. This can also be the impedance corresponding to a feedback system [4, 5]
β	Speed of the reference particle divided by c
$\beta_\perp$	Beta-function
γ	$1/\sqrt{1-\beta^2}$
Ω	Coherent frequency
$\boldsymbol{\omega}$	Incoherent oscillation frequencies

ω_0	Revolution frequency, equal to $2\pi\beta c/L$
ω_p	$p\omega_0 + \Omega$

Note that $q^2\beta N_n/\gamma m_q cL$ is the current in bunch n.

Equation (1) is valid under the following assumptions:

1. The lattice is linear, uncoupled, with dispersion ignored. One allowed exception is that first-order chromatic effects are treated by assuming that a constant times the longitudinal displacement is added to the transverse angle variables (this is the standard treatment of chromaticity [6, 7]).

2. The impedance is smooth, in the sense that if $Z(\omega)$ were redefined to include the dependence of the impedance on the position in the ring, the resulting function would be independent of s.

3. The buckets are spaced equally around the ring (although some buckets may be empty, corresponding to $N_n = 0$).

Allowing coupling or dispersion would probably not invalidate the following derivation, however.

At this point, ignore azimuthal mode coupling (i.e., coupling between modes with different $\boldsymbol{m}$). This will be a good approximation as long as the difference $\Omega - \boldsymbol{m}\cdot\boldsymbol{\omega}$ is small compared the synchrotron frequency. Then, defining $\Delta\nu = (\Omega - \boldsymbol{m}\cdot\boldsymbol{\omega})/\omega_0$, equation (1) can be written in block matrix form as

$$\Delta\nu\psi_r = \sum_n N_n A_{r-n}\psi_n, \tag{2}$$

where the vectors ψ_r have the a'th component $\Psi_{r\boldsymbol{m}a}$, and the components of the matrix A_k are

$$(A_k)_{ba} = -i\frac{q^2\beta}{\gamma m_q cL}\sum_p e^{2\pi ikp/M} Z(\omega_p) k_{\boldsymbol{m}b}(\omega_p) h_{\boldsymbol{m}a}(\omega_p). \tag{3}$$

Notice that A_k only depends on k mod M.

Note that A_k should also depend on Ω. So that (2) is a true eigenvalue system, it is useful to assume that A_k is independent of Ω. In practice, this independence only has to be over a scale comparable to the synchrotron frequency. This is satisfied in most cases, except when one has an extremely narrow-band impedance. Such a case arises when one looks at the fundamental mode of the r.f. cavities (i.e., for analysis of the Robinson instability). The corrections are typically small, even in this case, however.

3 Computation of the Eigenvalues

Now, consider an eigenvalue of the equation (2). First, consider the matrix B whose block components B_{rn} are A_{r-n}. The eigenvalues of B are the complex frequency shifts for the symmetric filling when every bucket contains one particle.

3.1 Eigenvalues of B

Consider the eigenvalue system $\lambda v = Bv$. Equation (3) implies that B_{rn} can be written as

$$B_{rn} = \sum_{p=0}^{M-1} e^{2\pi i(r-n)p/M} Z_p \qquad\qquad Z_p = \sum_{k=0}^{M-1} A_k e^{-2\pi ikp/M}. \tag{4}$$

The definition of Z_p implies that

$$(Z_p)_{ba} = -i\frac{q^2\beta}{\gamma m_q cL} \sum_{p'} Z(\omega_{p+p'M}) k_{\boldsymbol{m}b}(\omega_{p+p'M}) h_{\boldsymbol{m}a}(\omega_{p+p'M}). \tag{5}$$

If one does a change of basis to w such that

$$w_p = \sum_n e^{-2\pi inp/M} v_n, \tag{6}$$

then the eigenvalue equation simplifies greatly to the block-diagonal form $\lambda w_p = Z_p w_p$.

Thus, if one has truncated the radial mode expansion such that each vector v_n had dimension K, then an eigenvalue system of dimension MK has been reduced to M separate eigenvalue systems of size K (note that B has dimension MK, where A_k and Z_p have dimension K). This greatly simplifies the problem if M is large!

3.2 Normality of B

First, begin by defining the notation and terms that will be used subsequently. If v and w are vectors,

$$v^\dagger w = \sum_k v_k^* w_k, \tag{7}$$

where v_k is the k'th component of the vector v and v_k^* is its complex conjugate. Two vectors v and w are orthogonal if $v^\dagger w = 0$. Similarly, for a matrix A, $A^\dagger$ is the matrix whose components are given by $(A^\dagger)_{kl} = A_{lk}^*$.

Both the proof in [1] and this proof require that B be normal, which means that $B^\dagger B = BB^\dagger$, where $B^\dagger$ is the Hermitian conjugate of B. If B is normal and finite-dimensional, then this implies that B has a complete set of eigenvectors, and these eigenvectors are orthogonal [8].

So, compute $B^\dagger B$ and $BB^\dagger$:

$$(B^\dagger B)_{rn} = \sum_k A^\dagger_{k-r} A_{k-n} = \sum_k A^\dagger_{k+n-r} A_k \tag{8}$$

$$(BB^\dagger)_{rn} = \sum_k A_{r-k} A^\dagger_{n-k} = \sum_k A_k A^\dagger_{k+n-r}. \tag{9}$$

The second equality in each case takes advantage of the fact that A_k only depends on k mod M. Equations (8) and (9) imply that if A_k and $A^\dagger_l$ commuted, then B would be normal. However, in general they do not commute. If the matrices A_k were diagonal, however, then the commutation is trivial.

A_k being diagonal implies that coupling between radial modes (Ψ_{rmb} with different b) is ignored. Since these modes all have the same frequency as $N_n \to 0$, it is not clear that this coupling can be ignored. When the dominant frequencies in the impedance are small compared to $\beta c/\sigma_\ell$, where σ_ℓ is the bunch length, then one of the eigenvalues of Z_p will be much larger (in the sense of absolute value) than the others. The mode corresponding to this larger eigenvalue will be called the "lowest radial mode," whereas the others will be called "higher-order radial modes." Ignoring radial mode coupling will have a relatively small effect on the lowest radial mode (the correction will be comparable to the higher-order radial mode eigenfrequencies). However, it will have a strong effect on the higher-order radial modes.

In most cases, a well-chosen basis will make the lowest radial mode have one component dominant in its vector representation and the other components small, and this component will be the same for all p. Without loss of generality, this can be taken to be the first component. Then, in the matrix of Z_p, one only need zero the off-diagonal elements in the first row and column in making the approximation of ignoring radial mode coupling. To the extent that this does not perturb the eigenvalue for the lowest radial mode, the approximation is thus valid.

Thus, from this point B will be considered to be an $M \times M$ matrix, whose components are constructed by applying the definition of B and (4) to $Z_p = \lambda_p$, where λ_p is the eigenvalue for the lowest radial mode of symmetric mode p. Since Z_p is a scalar, so is A_k, and B is thus normal. Since B is finite-dimensional by construction, its eigenvectors are orthogonal [8]. v_p will be used to denote the orthonormal eigenvectors of B in what follows.

3.3 Eigenvalues for the General System

Call N the diagonal matrix whose entries are N_n. Then (2) can be written as $\Delta\nu\psi = BN\psi$. Performing a change of basis to $\phi = N^{1/2}\psi$, the eigenvalue equation is $\Delta\nu\phi = N^{1/2}BN^{1/2}\phi$. Assume that an eigenvalue $\Delta\nu$ has been found. There is always at least one eigenvector ϕ corresponding to that eigenvalue [8].

The eigenvalue can thus be written as

$$\Delta\nu = \frac{\phi^\dagger N^{1/2} B N^{1/2} \phi}{\phi^\dagger \phi}. \tag{10}$$

One can write $N^{1/2}\phi = \sum_p c_p v_p$, where the v_p are the basis functions described in the previous subsection. Thus, since B is normal, (10) becomes

$$\Delta\nu = \frac{\sum\limits_p |c_p|^2 \lambda_p}{\sum\limits_p |c_p|^2} \frac{\phi^\dagger N \phi}{\phi^\dagger \phi}. \tag{11}$$

The second fraction in (11) is a real number lying between the smallest N_n and the largest N_n. As for the first fraction,

$$\min_p \mathrm{Re}\{\lambda_p\} \leqslant \frac{\sum\limits_p |c_p|^2 \,\mathrm{Re}\{\lambda_p\}}{\sum\limits_p |c_p|^2} \leqslant \max_p \mathrm{Re}\{\lambda_p\} \tag{12}$$

$$\min_p \mathrm{Im}\{\lambda_p\} \leqslant \frac{\sum\limits_p |c_p|^2 \,\mathrm{Im}\{\lambda_p\}}{\sum\limits_p |c_p|^2} \leqslant \max_p \mathrm{Im}\{\lambda_p\}. \tag{13}$$

Thus,

$$\min_{pn}(\mathrm{Re}\{\lambda_p\} N_n) \leqslant \mathrm{Re}\{\Delta\nu\} \leqslant \max_{pn}(\mathrm{Re}\{\lambda_p\} N_n) \tag{14}$$

$$\min_{pn}(\mathrm{Im}\{\lambda_p\} N_n) \leqslant \mathrm{Im}\{\Delta\nu\} \leqslant \max_{pn}(\mathrm{Im}\{\lambda_p\} N_n). \tag{15}$$

4 Conclusions

The main result consists of equations (14) and (15). Note that the result applies individually for the lowest radial mode for each azimuthal mode $\boldsymbol{m}$.

In a real machine, the imaginary parts of the λ_p are both negative and positive. Thus, to find a bound on the growth rate

1. Find the bunch with the largest number of particles.
2. Solve the case with a symmetric filling assuming that all the buckets are filled with that number of particles.
3. Take the largest growth rate.

Because of the definition of the lowest radial mode, this result should apply to all of the higher radial modes as well.

In the presence of a feedback system which damps all the modes for a symmetric filling, all the modes will be damped in the non-symmetric filling, assuming

the feedback system has sufficient power to maintain the the same gain in either case. One must be cautious in applying the the definition of "lowest radial mode" in this case, since the feedback system can in principle zero the eigenvalue of what would be the lowest radial mode if there were no feedback (generally this is not the case: only the imaginary part is reduced, while the real part is uncorrected). A feedback system which truly nearly zeroed the eigenvalue of the lowest radial mode would probably invalidate this derivation.

The real parts of the λ_k all tend to have the same sign. The above argument gives a bound for the largest real frequency shift. It gives a bound on the smallest real frequency shift as well by

1. Finding the bunch with the smallest number of particles.
2. Solving the case with a symmetric filling assuming that all the buckets are filled with that number of particles.
3. Taking the smallest frequency shift.

In particular, if one or more buckets are empty, the shift could in principle be zero.

To summarize the assumptions required to reach this result:

1. Linear, uncoupled, dispersion-free lattice, with the exception of first-order chromaticity.
2. Smooth impedance.
3. The bunches are centered in buckets which are equally spaced around the ring.
4. Ignoring azimuthal mode coupling. This is a reasonable approximation if the complex frequency shifts are small compared to the synchrotron frequency.
5. Ignore radial mode coupling between the lowest radial mode and all other radial modes, and assume that a basis can be chosen such that the eigenvectors of the lowest radial mode all have the same dominant component. This will be a good approximation if the dominant frequencies in the impedance are low compared to the frequencies in the bunch spectrum.

In summary, this paper has demonstrated that one can find a bound on the growth rates of multibunch modes when the bunches have arbitrary amounts of current, subject to certain assumptions. This extends the work in [1], as well as elucidating some of the assumptions that are required to achieve either result.

References

[1] R. D. Kohaupt. On Multi-Bunch Instabilities for Fractionally Filled Rings. DESY-85-139 (1985).

[2] Gilbert Besnier. Stabilité des Oscillations Longitudinales d'un Faisceau Group se Propageant dans une Chambre a Vide d'Impedance Reactive. Nucl. Inst. Meth. **164** (1979), pp. 235-245.

[3] B. Zotter. Longitudinal Stability of Bunched Beams; Part I: Resonator Impedances. CERN SPS/81-18 (DI) (1981).

[4] J. Scott Berg. Coherent Modes for Multiple Non-Rigid Bunches in a Storage Ring. Ph.D. Thesis, Stanford University. Also SLAC-R-478 (1996).

[5] J. Scott Berg and Ronald D. Ruth. Transverse Instabilities for Multiple Non-rigid Bunches in a Storage Ring. Phys. Rev. E **52** (1995), pp. 2179-2182.

[6] F. J. Sacherer. Transverse Bunched Beam Instabilities—Theory, in "Proceedings of the IXth International Conference on High Energy Accelerators" (1974).

[7] Alexander Wu Chao. "Physics of Collective Beam Instabilities in High Energy Accelerators." John Wiley & Sons, New York (1993).

[8] See any advanced linear algebra book. In particular: Kenneth Hoffman and Ray Kunze. "Linear Algebra," second edition. Prentice-Hall, Englewood Cliffs, NJ (1971).